Hilbert Space

TLDR. Hilbert space are just complete inner product spaces. The completenes just makes the infinite dimensional vector space similar to finite dimensional vector spaces, and the inner product is a technical condition that makes the theory much easier.

§ Hilbert space:

a complete inner product space. It's a vector space V with an inner product $g(-,-)$ on it (hence a norm) such that V is complete as a metric space with the metric induced by g.

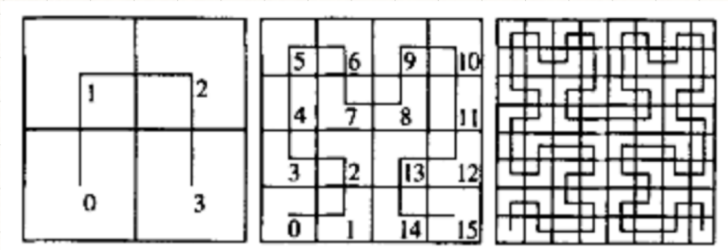
<Reference: Cornell math .edu>

Hilbert Curve — space-filling curve used for spatial indexing. (not to be confused with Hilbert space)

§ What's it?

Space-filling curve that visits every cell of $2^m \times 2^m$ grid exactly once while keeping good spatial locality

eg.



1st order 2nd order 3rd order

§ why it's useful?

- Map unordered 2D/3D points into 1D keys so neighbors stay near each other in memory
  → better cache locality, lower I/O

Hilbert Spaces

Recall that any inner product space V has an associated norm defined by

$$\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}.$$

Thus an inner product space can be viewed as a special kind of normed vector space. In particular, every inner product space V has a metric defined by

$$d(\mathbf{v}, \mathbf{w}) = \|\mathbf{v} - \mathbf{w}\| = \sqrt{\langle \mathbf{v} - \mathbf{w}, \mathbf{v} - \mathbf{w} \rangle}.$$

Definition: Hilbert space
A **Hilbert space** is an inner product space whose associated metric is complete.

That is, a Hilbert space is an inner product space that is also a Banach space. For example, $\mathbb{R}^n$ is a Hilbert space under the usual dot product:

$$\langle \mathbf{v}, \mathbf{w} \rangle = \mathbf{v} \cdot \mathbf{w} = v_1 w_1 + \cdots + v_n w_n.$$

More generally, a finite-dimensional inner product space is a Hilbert space. The following theorem provides examples of infinite-dimensional Hilbert spaces.

Theorem 1 L^2 is a Hilbert Space

For any measure space (X, μ), the associated L^2-space $L^2(X)$ forms a Hilbert space under the inner product

$$\langle f, g \rangle = \int_X f g \, d\mu.$$

PROOF The norm associated to the given inner product is the L^2-norm:

$$\|f\| = \sqrt{\langle f, f \rangle} = \sqrt{\int_X f^2 \, d\mu} = \|f\|_2.$$

We have already proven that $L^2(X)$ is complete with respect to this norm, and hence $L^2(X)$ is a Hilbert space. ■

In the case where $X = \mathbb{N}$, this gives us the following.

Corollary 2 ℓ^2 is a Hilbert Space

The space ℓ^2 of all square-summable sequences is a Hilbert space under the inner product

$$\langle \mathbf{v}, \mathbf{w} \rangle = \sum_{n \in \mathbb{N}} v_n w_n.$$

ℓ^2-Linear Combinations

We now turn to some general theory for Hilbert spaces. First, recall that two vectors $\mathbf{v}$ and $\mathbf{w}$ in an inner product space are called **orthogonal** if $\langle \mathbf{v}, \mathbf{w} \rangle = 0$.

Proposition 3 Convergence of Orthogonal Series

Let $\{\mathbf{v}_n\}$ be a sequence of orthogonal vectors in a Hilbert space. Then the series

$$\sum_{n=1}^{\infty} \mathbf{v}_n$$

converges if and only if

$$\sum_{n=1}^{\infty} \|\mathbf{v}_n\|^2 < \infty.$$

PROOF Let $\mathbf{s}_n$ be the sequence of partial sums for the given series. By the Pythagorean theorem,

$$\|\mathbf{s}_i - \mathbf{s}_j\|^2 = \left\| \sum_{n=i+1}^{j} \mathbf{v}_n \right\|^2 = \sum_{n=i+1}^{j} \|\mathbf{v}_n\|^2.$$

for all $i \leq j$. It follows that $\{\mathbf{s}_n\}$ is a Cauchy sequence if and only if $\sum_{n=1}^{\infty} \|\mathbf{v}_n\|^2$ converges. ■

We wish to apply this proposition to linear combinations of orthonormal vectors. First recall that a sequence $\{\mathbf{u}_n\}$ of vectors in an inner product space is called **orthonormal** if

$$\langle \mathbf{u}_i, \mathbf{u}_j \rangle \;=\; \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases}$$

for all i and j.

Corollary 4 ℓ^2-Linear Combinations

Let $\{\mathbf{u}_n\}$ be an orthonormal sequence of vectors in a Hilbert space, and let $\{a_n\}$ be a sequence of real numbers. Then the series

$$\sum_{n=1}^{\infty} a_n \mathbf{u}_n$$

converges if and only if the sequence $\{a_n\}$ lies in ℓ^2.

In general, if $\{a_n\}$ is an ℓ^2 sequence, then the sum

$$\sum_{n=1}^{\infty} a_n \mathbf{u}_n$$

is called a **ℓ^2-linear combination** of the vectors $\{\mathbf{u}_n\}$. By the previous corollary, every ℓ^2-linear combination orthonormal vectors in a Hilbert space converges

Proposition 5 Inner Product Formula

Let $\{\mathbf{u}_n\}$ be an orthonormal sequence of vectors in a Hilbert space, and let

$$\mathbf{v} \;=\; \sum_{n=1}^{\infty} a_n \mathbf{u}_n \qquad \text{and} \qquad \mathbf{w} \;=\; \sum_{n=1}^{\infty} b_n \mathbf{u}_n$$

be ℓ^2-linear combinations of the vectors $\{\mathbf{u}_n\}$. Then

$$\langle \mathbf{v}, \mathbf{w} \rangle \;=\; \sum_{n=1}^{\infty} a_n b_n.$$

PROOF Let $\mathbf{s}_N = \sum_{n=1}^{N} a_n \mathbf{u}_n$ and $\mathbf{t}_N = \sum_{n=1}^{N} b_n \mathbf{u}_n$, and note that $\mathbf{s}_N \to \mathbf{v}$ and $\mathbf{t}_N \to \mathbf{w}$ as $N \to \infty$. Since the inner product $\langle -, - \rangle$ is a continuous function, it follows that

$$\langle \mathbf{v}, \mathbf{w} \rangle = \lim_{N\to\infty} \langle \mathbf{s}_N, \mathbf{t}_N \rangle = \lim_{N\to\infty} \sum_{n=1}^{N} a_n b_n = \sum_{n=1}^{\infty} a_n b_n. \qquad \blacksquare$$

In the case where $\mathbf{v} = \mathbf{w}$, this gives the following.

Corollary 6 Norm Formula

Let $\{\mathbf{u}_n\}$ be an orthonormal sequence of vectors in a Hilbert space, and let

$$\mathbf{v} = \sum_{n=1}^{\infty} a_n \mathbf{u}_n$$

be an ℓ^2-linear combination of these vectors. Then

$$\|\mathbf{v}\| = \sqrt{\sum_{n=1}^{\infty} a_n^2}.$$

We can also use the inner product formula to find a nice formula for the coefficients of an ℓ^2-linear combination.

Corollary 7 Formula for the Coefficients

Let $\{\mathbf{u}_n\}$ be an orthonormal sequence of vectors in a Hilbert space, and let

$$\mathbf{v} = \sum_{n=1}^{\infty} a_n \mathbf{u}_n$$

be an ℓ^2-linear combination of these vectors. Then for all $n \in \mathbb{N}$,

$$a_n = \langle \mathbf{u}_n, \mathbf{v} \rangle.$$

PROOF Given an $n \in \mathbb{N}$, we can write $\mathbf{u}_n = \sum_{k=1}^{\infty} b_k \mathbf{u}_k$, where $b_n = 1$ and $b_k = 0$

for all $k \neq n$. By the inner product formula, it follows that

$$\langle \mathbf{u}_n, \mathbf{v} \rangle = \sum_{k=1}^{\infty} a_k b_k = a_n. \qquad \blacksquare$$

In general, we say that a vector $\mathbf{v}$ is in the **ℓ^2-span** of $\{\mathbf{u}_n\}$ if $\mathbf{v}$ can be expressed as an ℓ^2-linear combination of the vectors $\{\mathbf{u}_n\}$. According to the previous corollary, any vector $\mathbf{v}$ in the ℓ^2-span of $\{\mathbf{u}_n\}$ can be written as

$$\mathbf{v} = \sum_{n=1}^{\infty} \langle \mathbf{u}_n, \mathbf{v} \rangle \mathbf{u}_n.$$

It follows that

$$\|\mathbf{v}\| = \sqrt{\sum_{n=1}^{\infty} \langle \mathbf{u}_n, \mathbf{v} \rangle^2}$$

and

$$\langle \mathbf{v}, \mathbf{w} \rangle = \sum_{n=1}^{\infty} \langle \mathbf{u}_n, \mathbf{v} \rangle \langle \mathbf{u}_n, \mathbf{w} \rangle$$

for any two vectors $\mathbf{v}$ and $\mathbf{w}$ in the ℓ^2-span of $\{\mathbf{u}_n\}$.

Projections

> **Definition: Projection Onto a Subspace**
> Let V be an inner product space, let S be a linear subspace of V, and let $\mathbf{v} \in V$. A vector $\mathbf{p} \in S$ is called the **projection of v onto *S*** if
>
> $$\langle \mathbf{s}, \mathbf{v} - \mathbf{p} \rangle = 0$$
>
> for all $\mathbf{s} \in S$.

It is easy to see that the projection $\mathbf{p}$ of $\mathbf{v}$ onto S, if it exists, must be unique. In particular, if $\mathbf{p}_1$ and $\mathbf{p}_2$ are two possible projections, then

$$\|\mathbf{p}_1 - \mathbf{p}_2\|^2 = \langle \mathbf{p}_1 - \mathbf{p}_2, \mathbf{p}_1 - \mathbf{p}_2 \rangle = \langle \mathbf{p}_1 - \mathbf{p}_2, \mathbf{v} - \mathbf{p}_2 \rangle - \langle \mathbf{p}_1 - \mathbf{p}_2, \mathbf{v} - \mathbf{p}_1 \rangle,$$

and both of the inner products on the right are zero since $\mathbf{p}_1 - \mathbf{p}_2 \in S$.

It is always possible to project onto a finite-dimensional subspace.

Proposition 8 Projection Onto Finite-Dimensional Subspaces

Let V be an inner product space, let S be a finite-dimensional subspace of V, and let $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ be an orthonormal basis for S. Then for any $\mathbf{v} \in V$, the vector

$$\mathbf{p} = \sum_{k=1}^{n} \langle \mathbf{u}_k, \mathbf{v} \rangle \mathbf{u}_k$$

is the projection of $\mathbf{v}$ onto S.

PROOF Observe that $\langle \mathbf{u}_k, \mathbf{p} \rangle = \langle \mathbf{u}_k, \mathbf{v} \rangle$ for each k, and hence $\langle \mathbf{u}_k, \mathbf{v} - \mathbf{p} \rangle = 0$ for each k. By linearity, it follows that $\langle \mathbf{s}, \mathbf{v} - \mathbf{p} \rangle = 0$ for all $\mathbf{s} \in S$, and hence $\mathbf{p}$ is the projection of $\mathbf{v}$ onto S. ■

Our goal is to generalize this proposition to the ℓ^2-span of an orthonormal sequence.

Lemma 9 Bessel's Inequality

Let V be a Hilbert space, let $\{\mathbf{u}_n\}$ be an orthonormal sequence in V, and let $\mathbf{v} \in V$. Then

$$\sum_{n=1}^{\infty} \langle \mathbf{u}_n, \mathbf{v} \rangle^2 \leq \|\mathbf{v}\|^2.$$

PROOF Let $N \in \mathbb{N}$, and let

$$\mathbf{p}_N = \sum_{n=1}^{N} \langle \mathbf{u}_n, \mathbf{v} \rangle \mathbf{u}_n$$

be the projection of $\mathbf{v}$ onto $\mathrm{Span}\{\mathbf{u}_1, \ldots, \mathbf{u}_N\}$. Then $\langle \mathbf{p}_N, \mathbf{v} - \mathbf{p}_N \rangle = 0$, so by the Pythagorean theorem

$$\|\mathbf{v}\|^2 = \|\mathbf{p}_N\|^2 + \|\mathbf{v} - \mathbf{p}_N\|^2 \geq \|\mathbf{p}_N\|^2 = \sum_{n=1}^{N} \langle \mathbf{u}_n, \mathbf{v} \rangle^2.$$

This holds for all $N \in \mathbb{N}$, so the desired inequality follows. ■

Proposition 10 Projection Formula

Let V be a Hilbert space, and let $\{\mathbf{u}_n\}$ be an orthonormal sequence of vectors in V. Then for any $\mathbf{v} \in V$, the sequence $\{\langle \mathbf{u}_n, \mathbf{v}\rangle\}$ is ℓ^2, and the vector

$$\mathbf{p} = \sum_{n=1}^{\infty} \langle \mathbf{u}_n, \mathbf{v}\rangle \mathbf{u}_n$$

is the projection of $\mathbf{v}$ onto the ℓ^2-span of $\{\mathbf{u}_n\}$.

PROOF Bessel's inequality shows that the sequence $\{\langle \mathbf{u}_n, \mathbf{v}\rangle\}$ is ℓ^2, and thus the sum for $\mathbf{p}$ converges. By the coefficient formula (Corollary 7), we have that

$$\langle \mathbf{u}_n, \mathbf{p}\rangle = \langle \mathbf{u}_n, \mathbf{v}\rangle$$

for all $n \in \mathbb{N}$, and hence $\langle \mathbf{u}_n, \mathbf{v} - \mathbf{p}\rangle = 0$ for all $n \in \mathbb{N}$. By the continuity of $\langle -, -\rangle$, it follows that $\langle \mathbf{s}, \mathbf{v} - \mathbf{p}\rangle = 0$ for any $\mathbf{s}$ in the ℓ^2-span of $\{\mathbf{u}_n\}$, and hence $\mathbf{p}$ is the projection of $\mathbf{v}$ onto this subspace. ■

Hilbert Bases

Definition: Hilbert Basis
Let V be a Hilbert space, and let $\{\mathbf{u}_n\}$ be an orthonormal sequence of vectors in V. We say that $\{\mathbf{u}_n\}$ is a **Hilbert basis** for V if for every $\mathbf{v} \in V$ there exists a sequence $\{a_n\}$ in ℓ^2 so that

$$\mathbf{v} = \sum_{n=1}^{\infty} a_n \mathbf{u}_n.$$

That is, $\{\mathbf{u}_n\}$ is a Hilbert basis for V if every vector in V is in the ℓ^2-span of $\{\mathbf{u}_n\}$. For convenience, we are requiring all Hilbert bases to be countably infinite, but in the more general theory of Hilbert spaces a Hilbert basis may have any cardinality.

Note that a Hilbert basis $\{\mathbf{u}_n\}$ for V is not actually a basis for V in the sense of linear algebra. In particular, if $\{a_n\}$ is any ℓ^2 sequence with infinitely many nonzero terms, then the vector

$$\sum_{n=1}^{\infty} a_n \mathbf{u}_n$$

cannot be expressed as a finite linear combination of Hilbert basis vectors. Of course, it is clearly much more useful to allow ℓ^2-linear combinations, and in the context of Hilbert spaces it is common to use the word **basis** to mean Hilbert basis, while a standard linear-algebra-type basis is referred to as a **Hamel basis**.

EXAMPLE 1 The Standard Basis for ℓ^2

Consider the following orthonormal sequence in ℓ^2:

$$\mathbf{e}_1 = (1,0,0,0,\ldots), \qquad \mathbf{e}_2 = (0,1,0,0,\ldots), \qquad \mathbf{e}_3 = (0,0,1,0,\ldots), \qquad \ldots$$

If $\mathbf{v} = (v_1, v_2, \ldots)$ is a vector in ℓ^2, it is easy to show that

$$\mathbf{v} = \sum_{n=1}^{\infty} v_n \mathbf{e}_n,$$

and therefore $\{\mathbf{e}_n\}$ is a Hilbert basis for ℓ^2. ■

This example is in some sense quite general, as shown by the following proposition.

Proposition 11 Isomorphism With ℓ^2

Let V be a Hilbert space, and suppose that V has a Hilbert basis $\{\mathbf{u}_n\}$. Then there exists an isometric isomorphism $T\colon \ell^2 \to V$ such that $T(\mathbf{e}_n) = \mathbf{u}_n$ for each n.

PROOF Define a function $T\colon \ell^2 \to V$ by

$$T(a_1, a_2, \ldots) = \sum_{n=1}^{\infty} a_n \mathbf{u}_n.$$

Clearly T is linear. Note also that T is a bijection, with inverse given by

$$T^{-1}(\mathbf{v}) = \big(\langle \mathbf{u}_1, \mathbf{v}\rangle, \langle \mathbf{u}_2, \mathbf{v}\rangle, \ldots\big),$$

and hence T is a linear isomorphism. Finally, we have

$$\big\|T(a_1, a_2, \ldots)\big\| = \left\|\sum_{n=1}^{\infty} a_n \mathbf{u}_n\right\| = \sqrt{\sum_{n=1}^{\infty} a_n^2} = \big\|(a_1, a_2, \ldots)\big\|_2$$

for all $(a_1, a_2, \ldots) \in \ell^2$, so T is isometric. ■

Proposition 12 Characterization of Hilbert Bases

Let V be a Hilbert space, and let $\{\mathbf{u}_n\}$ be an orthonormal sequence of vectors in V. Then the following are equivalent:

1. *The sequence $\{\mathbf{u}_n\}$ is a Hilbert basis for V.*

2. *The set of all finite linear combinations of elements of $\{\mathbf{u}_n\}$ is dense in V.*

3. *For every nonzero $\mathbf{v} \in V$, there exists an $n \in \mathbb{N}$ so that $\langle \mathbf{u}_n, \mathbf{v} \rangle \neq 0$.*

PROOF Let S be the set of all finite linear combinations of elements of $\{\mathbf{u}_n\}$, i.e. the linear span of $\{\mathbf{u}_n\}$. We prove that (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (1).

(1) $\Rightarrow$ (2) Suppose that $\{\mathbf{u}_n\}$ is a Hilbert basis, and let $\mathbf{v} \in V$. Then

$$\mathbf{v} = \sum_{n=1}^{\infty} a_n \mathbf{u}_n$$

for some ℓ^2 sequence $\{a_n\}$. Then $\mathbf{v}$ is the limit of the sequence of partial sums

$$\mathbf{s}_N = \sum_{n=1}^{N} a_n \mathbf{u}_n,$$

so $\mathbf{v}$ lies in the closure of S.

(2) $\Rightarrow$ (3) Suppose that S is dense in V, and let $\mathbf{v}$ be a nonzero vector in V. Let $\{\mathbf{s}_n\}$ be a sequence in S that converges to $\mathbf{v}$. Then there exists an $n \in \mathbb{N}$ so that $\|\mathbf{s}_n - \mathbf{v}\| < \|\mathbf{v}\|$, and it follows that

$$\langle \mathbf{s}_n, \mathbf{v} \rangle = \frac{\|\mathbf{s}_n\|^2 + \|\mathbf{v}\|^2 - \|\mathbf{s}_n - \mathbf{v}\|^2}{2} > \frac{\|\mathbf{s}_n\|^2}{2} \geq 0.$$

But since $\mathbf{s}_n \in S$, we know that $\mathbf{s}_n \in \mathrm{Span}\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ for some $k \in \mathbb{N}$, and it follows that $\langle \mathbf{u}_i, \mathbf{v} \rangle \neq 0$ for some $i \leq k$.

(3) $\Rightarrow$ (1) Suppose that condition (3) holds, let $\mathbf{v} \in V$, and let

$$\mathbf{p} = \sum_{n=1}^{\infty} \langle \mathbf{u}_n, \mathbf{v} \rangle \mathbf{u}_n$$

be the projection of $\mathbf{v}$ onto the ℓ^2-span of $\{\mathbf{u}_n\}$ (by Proposition 10). Then $\langle \mathbf{u}_n, \mathbf{p} - \mathbf{v} \rangle = 0$ for all $n \in \mathbb{N}$, so by condition (3) the vector $\mathbf{p} - \mathbf{v}$ must be zero. Then $\mathbf{v} = \mathbf{p}$, so $\mathbf{v}$ lies in the ℓ^2-span of $\{\mathbf{u}_n\}$, which proves that $\{\mathbf{u}_n\}$ is a Hilbert basis. ■

Fourier Series

The theory of Hilbert spaces lets us provide a nice theory for Fourier series on the interval $[-\pi, \pi]$. We begin with the following theorem.

Theorem 13 Density of Continuous Functions

For any closed interval $[a, b] \subseteq \mathbb{R}$, the continuous functions on $[a, b]$ are dense in $L^2\big([a, b]\big)$.

PROOF See Homework 7, Problem 2 for a proof in the L^1 case. The L^2 case is quite similar. ■

It follows that any closed subset of $L^2\big([a, b]\big)$ that contains the continuous functions must be all of $L^2\big([a, b]\big)$.

Theorem 14 The Fourier Basis

The sequence

$$\frac{1}{\sqrt{2\pi}}, \quad \frac{\cos x}{\sqrt{\pi}}, \quad \frac{\sin x}{\sqrt{\pi}}, \quad \frac{\cos 2x}{\sqrt{\pi}}, \quad \frac{\sin 2x}{\sqrt{\pi}}, \quad \frac{\cos 3x}{\sqrt{\pi}}, \quad \frac{\sin 3x}{\sqrt{\pi}}, \quad \ldots$$

is a Hilbert basis for $L^2\big([-\pi, \pi]\big)$.

PROOF It is easy to check that the given functions are orthonormal. Let S be the set of all finite linear combinations of the basis elements, i.e. the set of all finite trigonometric polynomials. By Proposition 12, it suffices to prove that S is dense in $L^2\big([-\pi, \pi]\big)$.

Let $C(T)$ be the set of all continuous functions f on $[-\pi, \pi] \to \mathbb{R}$ for which $f(-\pi) = f(\pi)$. By Homework 10, every function in $C(T)$ is the uniform limit (and hence the L^2 limit) of trigonometric polynomials, so the closure of S contiains $C(T)$. But clearly every continuous function on $[a, b]$ is the L^2 limit of functions in $C(T)$, and hence the closure of S contains every continuous function. By Theorem 13, we conclude that the closure of S is all of $L^2\big([-\pi, \pi]\big)$. ■

In general, an orthogonal sequence $\{f_n\}$ of nonzero L^2 functions on $[a, b]$ is called a

complete orthogonal system for $[a,b]$ if the sequence $\{f_n/\|f_n\|_2\}$ of normalizations is a Hilbert basis for $L^2([a,b])$. According to the above theorem, the sequence

$$1, \quad \cos x, \quad \sin x, \quad \cos 2x, \quad \sin 2x, \quad \cos 3x, \quad \sin 3x, \quad \ldots$$

is a complete orthogonal system for the interval $[-\pi,\pi]$.

Definition: Fourier Coefficients
Let $f\colon[-\pi,\pi]\to\mathbb{R}$ be an L^2 function. Then the **Fourier coefficients** of f are defined as follows:

$$a = \frac{\langle f,1\rangle}{2\pi} = \frac{1}{2\pi}\int_{[-\pi,\pi]} f\,dm,$$

$$b_n = \frac{\langle f,\cos nx\rangle}{\pi} = \frac{1}{\pi}\int_{[-\pi,\pi]} f(x)\cos nx\,dm(x),$$

$$c_n = \frac{\langle f,\sin nx\rangle}{\pi} = \frac{1}{\pi}\int_{[-\pi,\pi]} f(x)\sin nx\,dm(x).$$

Note that the Fourier coefficients are the coefficients for the functions

$$1, \quad \cos x, \quad \sin x, \quad \cos 2x, \quad \sin 2x, \quad \cos 3x, \quad \sin 3x, \quad \ldots,$$

which are not unit vectors. The actual coefficients of the Hilbert basis vectors are

$$a\sqrt{2\pi}, \qquad \{b_n\sqrt{\pi}\}, \qquad \text{and} \qquad \{c_n\sqrt{\pi}\}.$$

Corollary 15 Riesz-Fischer Theorem

Let $f\colon[-\pi,\pi]\to\mathbb{R}$ be an L^2 function with Fourier coefficients a, $\{b_n\}$ and $\{c_n\}$. Then $\{b_n\}$ and $\{c_n\}$ are ℓ^2 sequences, and the Fourier series

$$a + \sum_{n=1}^{\infty}\big(b_n\cos nx + c_n\sin nx\big)$$

converges to f in L^2.

PROOF This follows from Theorem 14 and the coefficient formula (Corollary 7). ■

Corollary 16 Parseval's Theorem

Let $f\colon [-\pi,\pi]\to\mathbb{R}$ be an L^2 function with Fourier coefficients a, $\{b_n\}$, $\{c_n\}$, and let $g\colon[-\pi,\pi]\to\mathbb{R}$ be an L^2 function with fourier coefficients A, $\{B_n\}$, and $\{C_n\}$. Then

$$\frac{1}{\pi}\int_{[-\pi,\pi]} fg\,dm \;=\; 2aA+\sum_{n=1}^{\infty}(b_nB_n+c_nC_n).$$

PROOF By the inner product formula (Proposition 5), we have

$$\langle f,g\rangle \;=\; \big(a\sqrt{2\pi}\big)\big(A\sqrt{2\pi}\big)+\sum_{n=1}^{\infty}\Big(\big(b_n\sqrt{\pi}\big)\big(B_n\sqrt{\pi}\big)+\big(c_n\sqrt{\pi}\big)\big(C_n\sqrt{\pi}\big)\Big),$$

and dividing through by π gives the desired formula. ∎

In the case where $g=f$, this theorem yields **Parseval's identity**:

$$\frac{1}{\pi}\int_{[-\pi,\pi]} f^2\,dm \;=\; 2a^2+\sum_{n=1}^{\infty}(b_n^2+c_n^2).$$

Corollary 17 Isomorphism of L^2 and ℓ^2

If $a<b$, then $L^2([a,b])$ and ℓ^2 are isometrically isomorphic.

PROOF Since

$$\frac{1}{\sqrt{2\pi}},\quad \frac{\cos x}{\sqrt{\pi}},\quad \frac{\sin x}{\sqrt{\pi}},\quad \frac{\cos 2x}{\sqrt{\pi}},\quad \frac{\sin 2x}{\sqrt{\pi}},\quad \frac{\cos 3x}{\sqrt{\pi}},\quad \frac{\sin 3x}{\sqrt{\pi}},\quad \ldots$$

is a Hilbert basis for $L^2\big([-\pi,\pi]\big)$, it follows from Proposition 11 that the linear transformation $T\colon \ell^2\to L^2\big([-\pi,\pi]\big)$ defined by

$$T(a_1,a_2,a_3,\ldots) \;=\; \frac{a_1}{\sqrt{2\pi}}+\frac{a_2\cos x}{\sqrt{\pi}}+\frac{a_3\sin x}{\sqrt{\pi}}+\frac{a_4\cos 2x}{\sqrt{\pi}}+\frac{a_5\sin 2x}{\sqrt{\pi}}+\cdots$$

is an isometric isomorphism. ∎

Other Orthogonal Systems

The Fourier basis is not the only Hilbert basis for $L^2\big([a,b]\big)$. Indeed, many such families of orthogonal functions are known. In this section, we derive an orthonormal sequence of polynomials that is a Hilbert basis for $L^2\big([a,b]\big)$.

Consider the sequence of functions

$$1, \quad x, \quad x^2, \quad x^3, \quad \ldots$$

on the interval $[-1,1]$. These functions are not a Hilbert basis for $L^2\big([-1,1]\big)$, since they are not orthonormal. However, it is possible to use these functions to make a Hilbert basis of polynomials via the **Gram-Schmidt process**. We start by making the the constant function 1 into a unit vector:

$$p_0(x) \;=\; \frac{1}{\|1\|_2} \;=\; \frac{1}{\sqrt{2}}.$$

The function x is already orthogonal to p_0 on the interval $[-1,1]$, so we normalize x as well:

$$p_1(x) \;=\; \frac{x}{\|x\|_2} \;=\; x\sqrt{\frac{3}{2}}.$$

Now we want a quadratic polynomial orthogonal to p_0 and p_1. The function x^2 is already orthogonal to p_1, but not to p_0. However, if we subtract from x^2 the projection of x^2 onto p_0, then we get a quadratic polynomial orthogonal to p_0:

$$x^2 - \langle p_0, x^2\rangle\, p_0(x) \;=\; x^2 - \frac{1}{3}.$$

Normalizing gives:

$$p_2(x) \;=\; \frac{3\sqrt{5}}{2\sqrt{2}}\left(x^2 - \frac{1}{3}\right)$$

Continuing in this fashion, we obtain an orthonormal sequence $\{p_n\}$ of polynomials, where each p_n is obtained from x^n by subtracting the projections of x^n onto $p_0, \ldots, p_{n-1}$ and then normalizing.

Definition: Legendre Polynomials
The **normalized Legendre polynomials** are the sequence of polynomial functions $p_n\colon [-1,1] \to \mathbb{R}$ defined recursively by $p_0(x) = 1/\sqrt{2}$ and

$$p_n(x) \;=\; c_n\left(x^n - \sum_{k=0}^{n-1} \langle p_k, x^n\rangle\, p_k(x)\right)$$

for $n \geq 1$, where the constant $c_n > 0$ is chosen so that $\|p_n\|_2 = 1$.

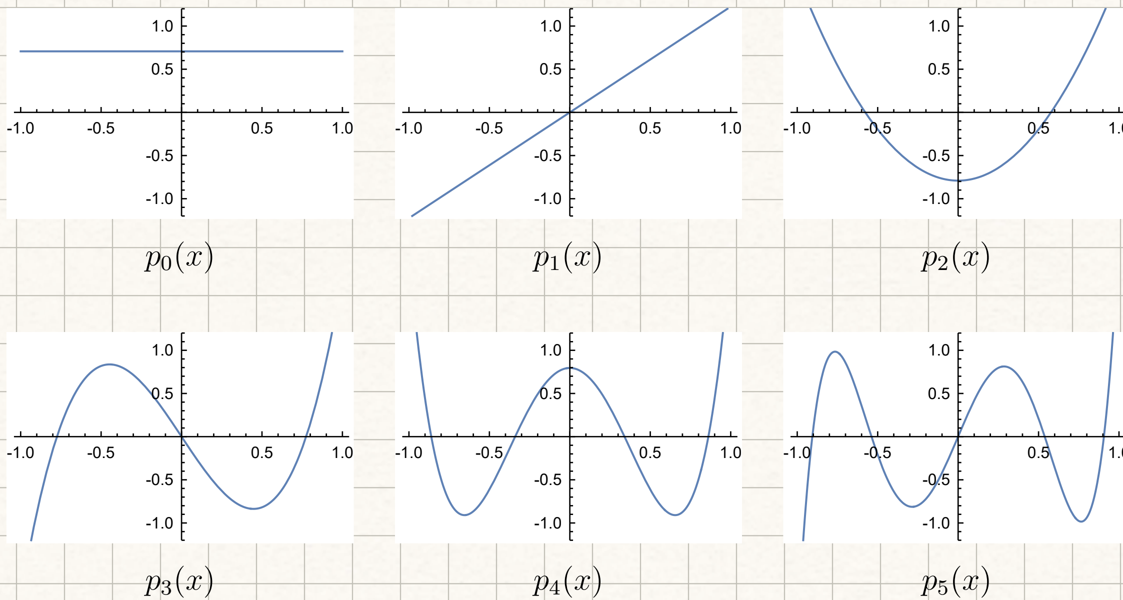


Figure 1: The normalized Legendre polynomials $p_0, \ldots, p_5$.

By design, each normalized Legendre polynomial $p_n(x)$ has degree n, and the sequence $\{p_n\}_{n \geq 0}$ is orthonormal. The next few such polynomials are

$$p_3(x) = \frac{5\sqrt{7}}{2\sqrt{2}}\left(x^3 - \frac{3}{5}x\right), \qquad p_4(x) = \frac{105}{8\sqrt{2}}\left(x^4 - \frac{6}{7}x^2 + \frac{3}{35}\right), \qquad \ldots$$

Figure 1 shows the graphs of the first six normalized Legendre polynomials.

Theorem 18 The Legendre Basis

The sequence $p_0, p_1, p_2, \ldots$ of normalized Legendre polynomials is a Hilbert basis for $L^2\big([-1,1]\big)$.

PROOF Let S be the linear span of $p_0, p_1, p_2, \ldots$. Since

$$x^n = \frac{p_n(x)}{c_n} + \sum_{k=0}^{n-1} \langle p_k, x^n \rangle \, p_k(x),$$

the subspace S contains each x^n, and hence contains all polynomials. By the Weierstrass approximation theorem, every continuous function on $[-1,1]$ is a uniform limit (and hence and L^2 limit) of a sequence of polynomials. It follows that the closure of S contains all the continuous functions, and hence contains all L^2 functions by Theorem 13. ∎

Thus every L^2 function f on $[-1,1]$ can be written as the sum of an infinite **Legendre series**

$$f = \sum_{n=0}^{\infty} \langle p_n, f\rangle p_n.$$

These behave much like Fourier series, with analogs of Parseval's theorem and Parseval's identity.

Legendre polynomials are important in partial differential equations. For the following definition, recall that a **harmonic function** on a closed region in $\mathbb{R}^3$ is any continuous function that satisfies Laplace's equation $\nabla^2 f = 0$ on the interior of the region.

Definition: Dirichlet Problem on a Ball
Let B^3 denote the closed unit ball on $\mathbb{R}^3$, and let S^2 denote the unit sphere. The **Dirichlet problem** on B^3 can be stated as follows:

> Given a continuous function $f\colon S^2 \to \mathbb{R}$, find a harmonic function $F\colon B^3 \to \mathbb{R}$ that agrees with f on S^2.

Since we are working on the ball, it makes sense to use **spherical coordinates** (ρ, θ, ϕ), which are defined by the formulas

$$x = \rho\cos\theta\sin\phi, \qquad y = \rho\sin\theta\sin\phi, \qquad z = \rho\cos\phi.$$

Using spherical coordinates, one family of solutions to Laplace's equation on the ball can be written as follows:

$$F(\rho,\theta,\phi) = \rho^n\, p_n(\cos\phi)$$

where p_n is the nth Legendre polynomial. These solutions are all **axially symmetric** around the z-axis, meaning that they have no explicit dependence on θ.

Since the Legendre polynomials are a Hilbert basis, we can use these solutions to solve the Dirichlet problem for any axially symmetric function $f\colon S^2 \to \mathbb{R}$. All we do is write f as the sum of a Legendre series

$$f(\theta,\phi) = \sum_{n=0}^{\infty} a_n\, p_n(\cos\phi),$$

and then the corresponding harmonic function F will be defined by the formula

$$F(\rho,\theta,\phi) = \sum_{n=0}^{\infty} a_n \rho^n p_n(\cos\phi).$$

EXAMPLE 1 Let $f\colon S^2 \to \mathbb{R}$ be the function defined by

$$f(x,y,z) \;=\; z^2.$$

Find a harmonic function $F\colon B^3 \to \mathbb{R}$ that agrees with f on S^2.

SOLUTION Note that $z = \cos\phi$ on S^2, so we can write f as

$$f(\theta,\phi) \;=\; \cos^2\phi.$$

Since

$$p_0(x) \;=\; c_0 \qquad \text{and} \qquad p_2(x) \;=\; c_2\left(x^2 - \tfrac{1}{3}\right)$$

where $c_0 = 1/\sqrt{2}$ and $c_2 = \sqrt{45/8}$, we can write f as

$$f(\theta,\phi) \;=\; \frac{1}{3c_0}p_0(\cos\phi) + \frac{1}{c_2}p_2(\cos\phi).$$

Then the corresponding harmonic function $F\colon B^3 \to \mathbb{R}$ is given by

$$F(\rho,\theta,\phi) \;=\; \frac{1}{3c_0}p_0(\cos\phi) + \frac{\rho^2}{c_2}p_2(\cos\phi) \;=\; \frac{1}{3} + \rho^2\left(\cos^2\phi - \frac{1}{3}\right). \qquad \blacksquare$$

The functions $p_n(\cos\phi)$ on the unit sphere can be generalized to the family of **spherical harmonics** $Y_{\ell,m}(\theta,\phi)$, which are a Hilbert basis for $L^2(S^2)$. The Legendre polynomials defined above correspond to the $m = 0$ case:

$$Y_{\ell,0}(\theta,\phi) \;=\; \frac{1}{\sqrt{2\pi}}\,p_\ell(\cos\phi).$$

Every L^2 function f on the sphere has a Fourier decomposition in terms of spherical harmonics:

$$f(\theta,\phi) \;=\; \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} a_{\ell,m}Y_{\ell,m}(\theta,\phi).$$

In quantum mechanics, these spherical harmonics give rise to the eigenfunctions of the square of the angular momentum operator. These are known as **atomic orbitals**, and can be used to describe the quantum wave functions of electrons in an atom.